Measuring Stochastic Long-Range Dependence
Calculating the Hurst Exponent of the S&P 500

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Outline

1 Economic Assumptions
   - Common Assumptions
   - Initial Analysis

2 Findings
   - Distributions
   - Calculations

3 Summary
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- With these findings, assuming common rationality, all asset prices reflect complete information, i.e. EMH (Fama)
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What Went Wrong?

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- Where do we go from here? Fractal analysis.
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Taking a look at real data, we’ll test this assumption and find a better fit.
Choosing a Distribution
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Cauchy Distribution

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- If we took the two most extreme events (The Emergency Banking Act increase and the Black Monday Crash), and sampled a random Gaussian change every second, we would expect both of these to occur approximately every $10^{101}$ years.
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- Thus, the Cauchy distribution is a better fit for the long, fat tailed data.
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Fractal Dimension

\[ C^d_H(S) := \inf \left\{ \sum_i r_i^d : \text{there is a cover of } S \text{ by balls with radii } r_i > 0 \right\}. \]

\[ \dim_H(X) := \inf \{ d \geq 0 : C^d_H(X) = 0 \} \]

Above, we have the definition of the \textbf{Hausdorff Dimension}, \( D \).
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Essentially, this dimension tells us how spaces scale
Scaling

S&P 500 Closing Price

YEAR

INDEX PRICE (USD)

'27 '30 '32 '35 '38 '41 '44 '47 '50 '53 '56 '59 '62 '65 '68 '71 '74 '77 '80 '83 '86 '89 '92 '95 '98 '01 '04 '07 '10 '13 '16 '19 '22 '25 '28 '31 '34 '37 '40 '43 '46 '49 '52 '55 '58 '61 '64 '67 '70 '73 '76 '79 '82 '85 '88 '91 '94 '97 '00 '03 '06 '09 '12 '15 '18 '21 '24 '27 '30 '33 '36 '39 '42 '45 '48 '51 '54 '57 '60 '63 '66 '69 '72 '75 '78 '81 '84 '87 '90 '93 '96 '99 '02 '05 '08 '11 '14 '17
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Curve Fitting

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INDEX PRICE (USD) 0 500 1000 1500 2000 2500 3000
Curve Fitting

Closing Price Hausdorff Dimension

Log(Curve Length) vs. Log(Population/Divisions)
Curve Fitting

Closing Price Hausdorff Dimension

Log(Curve Length) vs. Log(Population/Divisions)

Equation:

\[ y = -0.3553x + 4.907 \]

\[ R^2 = 0.9984 \]
Curve Fitting

High Price Hausdorff Dimension

\[ y = -0.2842x + 4.835 \]

\[ R^2 = 0.9963 \]
Curve Fitting

Low Price Hausdorff Dimension

\[ y = -0.3142x + 4.8854 \]

\[ R^2 = 0.9951 \]
Hausdorff Dimension and Hurst Exponent

- The closing price SP 500 has a Hausdorff dimension of 1.3553 ($D_{high} = 1.2849$, $D_{low} = 1.3142$)
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This value corresponds to a Hurst exponent of 0.6447 ($H_{high} = 0.7151$, $H_{low} = 0.6858$).
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- The closing price SP 500 has a Hausdorff dimension of 1.3553 ($D_{\text{high}} = 1.2849$, $D_{\text{low}} = 1.3142$)
- This value corresponds to a Hurst exponent of 0.6447 ($H_{\text{high}} = 0.7151$, $H_{\text{low}} = 0.6858$)
- Comparing this to the Hurst exponent value of the closing price computed by Bayraktar, et. al, of $0.6156 \pm 0.0531$, we see that these results are consistent
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- Increase the data set size to improve accuracy of these findings.
- Apply this curve fitting algorithm to different market data to determine better measures of volatility and risk.
B. Mandelbrot
*How Long Is the Coast of Britain? Statistical Self-Similarity and Fractional Dimension.*

E. Bayraktar, H. Poor, & K. Sircar
*Estimating the Fractal Dimension of the S&P 500 Index using Wavelet Analysis*

R. Hudson & B. Mandelbrot
The Misbehavior of Markets